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Conjugacy Separability of Certain 1-Relator Groups with Torsion

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A group A is said to be conjugacy separable (c.s.) if for each pair of elements $x, y \in A$ such that x, y are not conjugate in A there exists a finite homomorphic image $\bar{A}$ of A such that the images of x, y in $\bar{A}$ are not conjugate in $\bar{A}$. We show that certain special polyhedral groups and 1-relator groups with torsion of the form $\langle b, t; (t^{-1}b^l t b^m)^s\rangle$, $s > 1$, are c.s. The latter partly answers a question raised by Anshel in *Math. Rev.* 81f Review 20038.

1.

A group A is said to be conjugacy separable (c.s.) if for every pair of elements $x, y \in A$ such that x and y are not conjugate in A there exists a finite homomorphic image $\bar{A}$ of A such that the images of x, y in $\bar{A}$ are not conjugate in $\bar{A}$. This concept was first introduced by Blackburn [5] to solve the conjugacy problem for finitely generated (f.g.) nilpotent groups. Subsequently Mostowski [10] showed that finitely presented c.s. groups have solvable conjugacy problem. Thus he raised the question as to which groups are c.s. Partial results were obtained by Stebe for generalized free products (g.f.p.) of free groups with cyclic amalgamation [14] and Fuchsian groups [15]. Subsequently Dyer [6] showed that the g.f.p. of two free groups amalgamating a cyclic group is c.s. by first showing that the g.f.p. of two c.s. groups amalgamating a finite group is c.s. A parallel result by Collins (Theorem 13 in [6]) states that every *HNN* extension of a c.s. group with finite associated groups is c.s. This immediately leads to the question whether every *HNN* extension of a c.s. group with cyclic associated groups is c.s. The answer is clearly negative since the well-known Baumslag–Solitar group [4] is not even residually finite ($_R$F). However, in view of Baumslag's conjecture [2] that 1-relator groups with nontrivial torsion are $_R$F, and motivated by the above results of Collins and Dyer

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[6], one naturally asks whether 1-relator groups of the form $\langle b, t; (t^{-1}b^l t b^m)^s\rangle$, $s>1$, are c.s. In this paper we show that these groups are indeed c.s. In the course of the proof, Section 3, we need to show that certain polyhedral groups (which are degenerate cases of Fuchsian groups) are c.s. We raise the question whether all polyhedral groups are c.s. In view of Stebe's work [15] and that of the present paper, we further ask whether all Fuchsian groups and all 1-relator groups with nontrivial torsion are c.s. The answer to this last question is expected to be very hard since a proof of Baumslag's conjecture is still unknown while the conjugacy problem for 1-relator group with nontrivial torsion is always solvable (Newman [11]).

2. Preliminaries

Throughout this paper we shall let

$$G=\langle b, t; (t^{-1}b^l t b^m)^s\rangle, \qquad s>1,$$

$$B=\langle h, k; (h^m k^l)^s\rangle, \qquad s>1,$$

$$B_w=\langle h, k; (h^m k^l)^s = h^{lmw} = k^{lmw} = 1\rangle, \qquad s>1,$$

where w is a positive integer. [Note that we sometimes replace relators (e.g., $(h^m k^l)^s$) by relations (e.g., $(h^m k^l)^s=1$).] Clearly B_w is isomorphic to a factor group of B. By abuse of notation, we shall often use $\bar{x}$ to denote the image in B_w of x in B for each positive integer w, whilst retaining h and k as generators of B_w. H and K will denote the cyclic groups $\langle h\rangle$ and $\langle k\rangle$, respectively. It is well known that G is an HNN extension of the base group $B'=\langle b_0, b_1; (b_1^l b_0^m)^s\rangle$ by $\langle t\rangle$ with $b_{i+1}=t^{-1}b_i t$. Identifying h with b_0 and k with b_1, we see immediately B and B' are isomorphic. Thus we shall not attempt to distinguish the two groups. In the same spirit, we shall let

$$G_w=\langle h, k, t; (h^m k^l)^s = h^{lmw} = k^{lmw} = 1, t^{-1}ht = k\rangle.$$

Clearly G_w is an HNN extension of B_w by $\langle t\rangle$.

The following notations will be used for any group A:

$N \lhd_f A$ means N is a normal subgroup of finite index in A, $x \sim_A y$ means $x, y \in A$ and x is conjugate to y in A, $x \nsim_A y$ means x is not conjugate to y in A. If there is no confusion, we shall simply write $x \sim y$ and $x \nsim y$.

If A is an HNN extension of a base group R by $\langle t\rangle$ with associated subgroups L and M such that $t^{-1}Lt=M$ then $u \sim_{R,t^*} v$ will be used for $u, v \in R$ to mean either $u \sim_R v$ or $u\in L$ and $v=t^{-1}ut$ or $u \in M$ and $v=tut^{-1}$. $\|x\|$ will denote the usual normal form length of x in A if A is a g.f.p. or t-length of x if A is an HNN extension by $\langle t\rangle$, whichever is appropriate. (See Dyer [6].) x^A will denote the conjugacy class of x in A. $y \in x^A$ will be said to be of minimal length in its conjugacy class if there does not exists $z \in x^A$ such that $\|z\| < \|y\|$.

We refer the discussion of cyclically reduced words for g.f.p. and *HNN* extensions to either Dyer [6, pp. 37, 38] or Lyndon and Schupp [8, pp. 176, 185].

We recall the following terms in the literature:

x is *conjugacy distinguished* (c.d.) in A if, given $y \in A$ such that $x \not\sim y$ in A then there exists $N \lhd_f A$ such that $xN \not\sim yN$ in A/N (Stebe [14]).

Let L be a subgroup of A. Then A is said to be *L-separable* if for each $x \in A \backslash L$ there exists $N \lhd_f A$ such that $xN \notin LN/N$. In particular if A is L-separable for all cyclic subgroups L of A then A is said to be Π_c (Stebe [13]).

A is said to be $\langle x \rangle$*-potent*, briefly $\langle x \rangle$-pot [1], if to each positive integer n, there exists $N \lhd_f A$ such that xN has order exactly n in A/N. G is said to be *potent* if A is $\langle x \rangle$-pot for all $1 \neq x \in A$.

If A is a g.f.p. and x is written in normal (alternating) form, we shall use init x to denote the initial letter of x.

The basis of our proof rests on the following theorems:

THEOREM A (Dyer, Theorem 4, [6]). *If the groups X and Y are c.s. and U is finite, then $X *_U Y$ is c.s.*

THEOREM B (Collins, Theorem 13, [6]). *Let A be c.s. and L, M be finite isomorphic subgroups. Then the HNN extension $\langle A, t;\ \mathrm{rel}\, A, t^{-1}Lt = M \rangle$ is c.s.*

THEOREM C (Tang, Theorem 3.5, [16]) *$B = \langle h, k; (h^m k^l)^s \rangle$ is c.s.*

So given $x, y \in G$ such that $x \not\sim_G y$, our aim is to find an integer w such that $\bar{x} \not\sim_{G_w} \bar{y}$ from which we deduce that G is c.s. by showing that B_w is c.s.

Applying Lemmas 3.1, 3.2, 3.3 of Allenby and Tang [1], we immediately have

LEMMA 2.1. *B is $\langle h^m \rangle$-pot and $\langle k^l \rangle$-pot.*

Applying the results of Stebe [13] we show that:

LEMMA 2.2. *B is Π_c.*

Proof. Let $M = \langle u, v; (uv)^s \rangle$, $s > 1$. Then $M = \langle u \rangle * \langle uv; (uv)^s \rangle$. Since both of the free factors are Π_c, Theorem 5 (Stebe [13]) tells us M is Π_c. Now $B = \langle h \rangle *_{h^m = u} M *_{v = k^l} \langle k \rangle$. Thus by two applications of Corollary 3.3 (Stebe [13]), it follows that B is Π_c.

Throughout this paper we shall freely make use of Solitar's theorem (Theorem 2, [6]), and Collins' lemma (Theorem 3, [6]). In the case of Collins' lemma, if $x, y \in P$, where P is an HNN extension of A by $\langle t \rangle$ and x, y are cyclically reduced then we shall express x, y in the following forms;

$$\begin{aligned} x &= t^{e(1)}u_1 t^{e(2)}u_2 \cdots t^{e(\lambda)}u_\lambda, \\ y &= t^{f(1)}v_1 t^{f(2)}v_2 \cdots t^{f(\mu)}v_\mu, \end{aligned} \tag{2.1}$$

where $u_i, v_i \in A$ and $e(i), f(i) = \pm 1$.

3. Some Lemmas

In this section we shall prove some lemmas necessary for the proof of our main result.

Lemma 3.1. *Let $x \in B_w$ such that $x \sim_{B_w} h^i$ for some i, $0 < i < lmw$. Then,*

(i) *$x \nsim_{B_w} k^j$ for $0 < j < lmw$,*

(ii) *if $x \sim_{B_w} h^j$, $0 < j < lmw$, then $i = j$.*

Proof. (i) Suppose $x \sim h^i$ and $x \sim k^j$. Then $h^i \sim k^j$. Let $|h^i| = |k^j| = \alpha$. Clearly $\alpha \mid lmw$. Let p be a prime dividing α. Then $p \mid l$ or $p \mid mw$. If $p \mid l$ so that $l = cp$, then let $B_w(p) = \langle \hat{h}, \hat{k};\, (\hat{h}^m \hat{k}^l)^s = \hat{h}^{cmw} = \hat{k}^{lmw} = 1 \rangle$. Since $B_w(p)$ is a homomorphic image of B_w under the obvious homomorphism, $\hat{h}^i \sim \hat{k}^j$. But $|\hat{h}^i| = \alpha/p$ and $|\hat{k}^j| = \alpha$ which contradicts $\hat{h}^i \sim \hat{k}^j$. Hence $h^i \nsim k^j$. Similarly if $p \mid mw$, so that $mw = dp$, then by considering $B'_w(p) = \langle \bar{h}, \bar{k};\, (\bar{h}^m \bar{k}^l)^s = \bar{h}^{lmw} = \bar{k}^{ld} = 1 \rangle$ we have $h^i \nsim_{B_w} k^j$.

(ii) Clearly $h^i \sim_{B_w} h^j$. Now,

$$B_w = \langle k; k^{lmw} \rangle \underset{k^l = v}{*} \langle h, v;\, (h^m v)^s = h^{lmw} = v^{mw} = 1 \rangle.$$

By (i), $h^i \nsim v^j$. This implies that h^i is of minimal length in its conjugacy class in B_w. Thus, by Solitar's theorem (Theorem 2, [6]) $h^i \sim_C h^j$, where $C = \langle h, v;\, (h^m v)^s = h^{lmw} = v^{mw} = 1 \rangle$. Now $C = \langle u, v;\, (uv)^s = u^{lw} = v^{mw} = 1 \rangle *_{u = h^m} \langle h; h^{lmw} \rangle$. Let $D = \langle u, v;\, (uv)^s = u^{lw} = v^{mw} = 1 \rangle$ and $H = \langle h; h^{lmw} \rangle$. Suppose $h^i \in H \setminus \langle u \rangle$. Let $\theta\colon C \to \bar{C}$ be such that $\ker \theta = \langle h^m \rangle^C$. If $h^i \sim_C u^\beta$ then $h^i\theta \neq 1$ is conjugate to 1 in $C\theta$. This is impossible. Hence $h^i \nsim_C u^\beta$. Thus h^i is of minimal length in its conjugacy class in C. Again by Solitar's theorem $h^i \sim_C h^j$ implies $h^i \sim_H h^j$. But this implies $i \equiv j \bmod lmw$. Hence we need only consider the case when $h^i, h^j \in \langle u \rangle$. Let $c \in C$ such that $h^i = c^{-1} h^j c$. Without loss of generality, we can assume $\|c\| \geqslant 1$ and $\operatorname{init} c = d_1 \in D \setminus \langle u \rangle$. By a length argument, we must have $d_1^{-1}h^j d_1 \in \langle u \rangle$. Thus it follows from Corollary 2 of [7] that $d_1 \in \langle u \rangle$ contradicting $d_1 \notin \langle u \rangle$. This proves the lemma.

LEMMA 3.2. *Let $x, y \in B$ such that $x \not\sim_{t*} y$ in G. Then there exists an integer w such that $\bar{x} \not\sim_{\bar{t}*} \bar{y}$ in G_w.*

Proof. The case where either x or y is 1 is trivial since G is ${}_{\mathrm{R}}\mathrm{F}$ (Allenby and Tang [1]). If $x \notin H \cup K$ then, since by Lemma 2.2 B is Π_c, we can pass, via a finite image $\bar{B}$ of B, to a suitable B_w such that $\bar{x} \notin \bar{H} \cup \bar{K}$ in B_w. Similarly if $y \notin H \cup K$. Clearly $\bar{x} \not\sim_{\bar{t}*} \bar{y}$ in G_w for such $\bar{x}, \bar{y}$ and so we may assume that $x, y \in H \cup K$. Again, if $x, y \in H$ (so that $x, y \notin K$ since $H \cap K = \langle 1 \rangle$), then there exists w such that $\bar{x}, \bar{y} \in \bar{H}$ and $\bar{x}, \bar{y} \notin \bar{K}$ in B_w. Similarly, if $x, y \in K$. Hence in these two cases too $\bar{x} \not\sim_{\bar{t}*} \bar{y}$ in G_w. Thus we need only consider the cases where

(i) $x \in H$, $y \in K$ and

(ii) $x \in K$, $y \in H$ (and, of course, $x \not\sim_{t*} y$ in G).

Case (i). Let $x = h^i$ and $y = k^j$. Since $x \not\sim_{t*} y$ it follows that $i \neq j$. Thus we can choose $w > |i| + |j|$. Then $i \not\equiv j \bmod w$. Hence $\bar{h}^i \not\sim_{\bar{t}*} \bar{k}^j$ in G_w.

Case (ii). Similar.

Our next lemma shows that if the conjugacy class of an element of B is disjoint from the subgroup H (or K) then, for some w, the conjugacy class of its image in B_w is disjoint from $\bar{H}$ (or $\bar{K}$).

LEMMA 3.3. *If $x \in B$ such that $x^B \cap H = \emptyset$ then there exists an integer w such that $\bar{x}^{B_w} \cap \bar{H} = \emptyset$ in B_w.*

Proof. Let $K = \langle k \rangle$ and $D = \langle h, v; (h^m v)^s \rangle$. Then $B = K *_{k^l = v} D$. Without loss of generality, we can assume that x is of minimal length in x^B.

Case 1. $\|x\| = 0$. Since $D = \langle h^m v; (h^m v)^s \rangle * \langle h \rangle$ we can clearly map B onto $\hat{B}_r = \hat{K} *_{\hat{k}^l = \hat{v}} \hat{D}_r$ where $\hat{K} = \langle \hat{k} \rangle$ is infinite cyclic and $\hat{D}_r = \langle \hat{h}^m \hat{v}; (\hat{h}^m \hat{v})^s \rangle * \langle \hat{h}; \hat{h}^{lmr} \rangle = \langle \hat{h}, \hat{v}; (\hat{h}^m \hat{v})^s = \hat{h}^{lmr} = 1 \rangle$. This implies $\hat{x} \in \langle \hat{v} \rangle$ is of infinite order. Since $\hat{h}$ is of finite order, it follows that $\hat{x}^{\hat{D}_r} \cap \hat{H} = \emptyset$ in $\hat{D}_r$. By Theorem A, $\hat{D}_r$ is c.s. and, by Theorem 5 of [13], $\hat{D}_r$ is Π_c. Thus, since $|\hat{H}|$ is finite, there exists a finite homomorphic image $\tilde{D}_r$ of $\hat{D}_r$ such that $\tilde{x}^{\tilde{D}_r} \cap \tilde{H} = \emptyset$ and such that $\tilde{h}^i \in \tilde{D}_r \backslash \langle \tilde{v} \rangle$ for $i = 1, ..., lmr - 1$. Let $\tilde{v}$ be of order t in $\tilde{D}_r$ and let $\tilde{B} = \langle \tilde{k}; \tilde{k}^{lt} \rangle *_{\tilde{k}^l = \tilde{v}} \tilde{D}_r$. Since $\tilde{x}$ commutes with $\tilde{k}$, $\tilde{x}^{\tilde{B}} \cap \tilde{H} = \emptyset$ in $\tilde{B}$. Now $\tilde{B}$ is c.s., whence $\tilde{B}$ has a finite image B' in which $x'^{B'} \cap H' = \emptyset$ in B'. It follows that there exists a suitable integer w such that $\bar{x}^{B_w} \cap \bar{H} = \emptyset$ in B_w.

Case 2. $\|x\| = 1$. This means $x \in K \backslash \langle v \rangle$ or $x \in D \backslash \langle v \rangle$. If $x \in K \backslash \langle v \rangle$, consider $\tilde{B} = B/D^B$. Clearly $\tilde{x}^{\tilde{B}} \cap \tilde{H} = \varnothing$ in $\tilde{B}$. Hence there exists an integer w such that $\bar{x}^{B_w} \cap \bar{H} = \varnothing$ in B_w. Suppose $x \in D \backslash \langle v \rangle$. Since $D = \langle h^m v; (h^m v)^s \rangle * H$ and $x^B \cap H = \varnothing$ we can assume, taking x to be of minimal length in x^D, that either (i) x has finite order dividing s or that (ii) $\|x\| > 1$ in D. In each case we can find an integer r such that $\hat{x}^{\hat{D}_r} \cap \hat{H} = \varnothing$ in $\hat{D}_r = \langle \hat{h}, \hat{v}; (\hat{h}^m \hat{v})^s = \hat{h}^{lmr} = 1 \rangle$. Now $\hat{D}_r$ is c.s. and $\hat{H}$ is finite. Hence there exists a finite homomorphic image $\tilde{D}_r$ of $\hat{D}_r$ such that $\tilde{x}^{\tilde{D}_r} \cap \tilde{H} = \varnothing$. If $\tilde{x} \in \langle \tilde{v} \rangle$ we can proceed as in Case 1. If $\tilde{x} \in \tilde{D}_r \backslash \langle v \rangle$ we find immediately that $\tilde{x}^{\tilde{B}_r} \cap \tilde{H} \neq \varnothing$ implies $\tilde{x}^{\tilde{D}_r} \cap \tilde{H} \neq \varnothing$. We now obtain B' and then B_w as in Case 1.

Case 3. $\|x\| \geqslant 2$. Since D is Π_c and D is $\langle v \rangle$-pot it follows that for some r we can map B onto $\tilde{B} = \tilde{K} *_{\tilde{k}^l = \tilde{v}} \tilde{D}$, where $\tilde{K} = \langle \tilde{k}; \tilde{k}^{lmr} \rangle$ and $\tilde{D}$ is finite so that $\|\tilde{x}\|$ in $\tilde{B}$ is the same as $\|x\|$ in B. Suppose $\tilde{y} \in \tilde{x}^{\tilde{B}}$ is of minimal length in $\tilde{x}^{\tilde{B}}$. Since x is cyclically reduced in B we must have $\tilde{x}$ is cyclically reduced in $\tilde{B}$. Hence by Solitar's theorem (Theorem 2, [6]), $\|\tilde{y}\| = \|\tilde{x}\| \geqslant 2$. This implies $\tilde{y} \notin \tilde{H} \subseteq \tilde{D}$. Thus $\tilde{x}^{\tilde{B}} \cap \tilde{H} = \varnothing$ in $\tilde{B}$. Now $\tilde{H}$ is finite and $\tilde{B}$ is c.s. It follows that there exists a finite homomorphic image $\hat{B}$ of B such that $\hat{x}^{\hat{B}} \cap \hat{H} = \varnothing$ in $\hat{B}$. Hence there exists an integer w such that $\bar{x}^{B_w} \cap \bar{H} = \varnothing$ in B_w.

Now, we shall show that if $1 \neq x \in B$ is conjugate to some element of H (or K) then x is not conjugate to any element of K (or H). Moreover x is not conjugate to any other element of H (or K).

LEMMA 3.4. *Let $1 \notin x \in B$ such that $x \sim_B h^i$ for some $i \neq 0$. Then $x \nsim_B k^j$ for all j and $x \notin_B h^j$ for $j \neq i$.*

Proof. Suppose $x \sim_B k^j$. Then $h^i \sim_B k^j$. Since $B = K *_{k^l = v} D$, where $D = \langle h, v; (h^m v)^s \rangle$, there exists a natural homomorphism of B onto $\hat{B}_r = \hat{K} *_{\hat{k}^l = \hat{v}} \hat{D}_r$, where $\hat{K} = \langle \hat{k} \rangle$ and $\hat{D}_r = \langle \hat{h}, \hat{v}; (\hat{h}^m \hat{v})^s = \hat{h}^{lmr} = 1 \rangle$. This implies $\hat{h}^i \sim_{\hat{B}_r} \hat{k}^j$. But this is impossible since $\hat{h}^i$ is of finite order in $\hat{B}_r$ and $\hat{k}_j$ is of infinite order. Hence $x \nsim_B k^j$.

Now, suppose $x \sim_B h^j$. Then $h^i \sim_B h^j$. If $|i| \neq |j|$ then w.l.o.g., we can assume $|i| < |j|$. Let $\hat{B}_j = \langle \hat{k} \rangle *_{\hat{k}^l = \hat{v}} \langle \hat{h}, \hat{v}; (\hat{h}^m \hat{v})^s = \hat{h}^{lmj} = 1 \rangle$. Then, as above, there exists a natural homomorphism of B onto $\hat{B}_j$. Thus $h^i \sim_B h^j$ implies that $\hat{h}^{lmi} \sim_{\hat{B}_j} \hat{h}^{lmj} = 1$. Since $|i| < |j|$, this is impossible, whence $h^i \nsim_B h^j$ for $|i| \neq |j|$. Therefore, we can assume $j = -i$. Now $x \nsim_B k^i$ implies that $x \nsim_B v^\alpha$. It follows that h^i is of minimal length in its conjugacy class in B. Moreover h^{-i} is clearly cyclically reduced. Thus, by Solitar's theorem, $h^i \sim_B h^{-i}$ implies that $h^i \sim_D h^j$. Let $\tilde{D} = D\theta$, where $h\theta = \tilde{h}$ and $v\theta = \tilde{h}^m$. Then $\tilde{D} = \langle \tilde{h} \rangle$ is an infinite cycle. Now $h^i \sim_D h^{-i}$ implies that $\tilde{h}^i \sim_{\tilde{D}} \tilde{h}^{-i}$. This clearly impossible. Hence the lemma is proved.

The next few technical lemmas prepare us to prove that if $y, z \in B$ such that $z \notin EyF$, where E, F are H or K, then there exists an integer w' such that $\bar{z} \notin \bar{E}\bar{y}\bar{F}$ in $B_{w'}$. We now write $B = H *_{h^m = u} D$ where $H = \langle h \rangle$ and $D = \langle u, k; (uk^l)^s \rangle$. We write H_m for $\langle u \rangle = \langle h^m \rangle$.

LEMMA 3.5. *Let $y, z \in B$ such that $z \notin KyH$ (or HyK). Let $N = D^B$ and $y = n_y h^i$, where $0 \leqslant i \leqslant m-1$ and $n_y \in N$. Let $u = h^m$. If $z \in N$ and there exists an integer w such that in B_w, $\bar{n}_y \bar{u}^\alpha \bar{z}^{-1} \notin \bar{K}$ for all α then $\bar{z} \notin \bar{K}\bar{y}\bar{H}$ in B_w.*

Proof. $\bar{n}_y \bar{u}^\alpha \bar{z}^{-1} \notin \bar{K}$ for all α if and only if $\bar{z} \notin \bar{K}\bar{n}_y \bar{u}$ for all α, that is, $\bar{z} \notin \bar{K}\bar{n}_y \bar{H}_m$ in B_w. Suppose $\bar{z} \in \bar{K}\bar{y}\bar{H}$ in B_w. This implies $\bar{z} \in \bar{K}\bar{n}_y\bar{H}$ in B_w. Thus there exists an integer $0 \leqslant i < m$ such that $\bar{z}\bar{h}^i \in \bar{K}\bar{n}_y\bar{H}_m$. Since $\bar{K}$, $\bar{n}_y$, $\bar{H}_m$ and $\bar{z}$ are all in $\bar{N}$, it follows that $\bar{h}^i \in \bar{N}$, whence $i = 0$. But this implies that $\bar{z} \in \bar{K}\bar{n}_y\bar{H}_m$ contradicting $\bar{z} \notin \bar{K}\bar{n}_y\bar{H}_m$. Hence $\bar{z} \notin \bar{K}\bar{y}\bar{H}$.

LEMMA 3.6. *Let $D_w = \langle u, k; (uk^l)^s = k^{lmw} = 1 \rangle$ and $w \geqslant 3$. Let $f, d \in D_w \backslash \langle u \rangle$ such that $fu^\alpha d = u^\beta$. If $fu^\delta d = u^\varepsilon$ then $\alpha = \delta$ and $\beta = \varepsilon$.*

Proof. $fu^\alpha d = u^\beta$ and $fu^\delta d = u^\varepsilon$ imply:

$$fu^{\alpha - \delta} f^{-1} = u^{\beta - \varepsilon}. \tag{3.6.1}$$

Now $D_w = \langle uk^l; (uk^l)^s \rangle * \langle k; k^{lmw} \rangle$. Consider $u = uk^l \cdot k^{-l}$ as a reduced word in the free decomposition of D_w. Then both $u^{\alpha-\delta}$ and $u^{\beta-\varepsilon}$ are cyclically reduced. Thus applying an easy length argument to (3.6.1) we must have $|\alpha - \delta| = |\beta - \varepsilon|$. Thus $\alpha - \delta = \pm(\beta - \varepsilon)$. Suppose $\beta - \varepsilon = -(\alpha - \delta)$. Then from (3.6.1) we have

$$(fuf^{-1})^{\alpha - \delta} = fu^{\alpha - \delta} f^{-1} = u^{-(\alpha - \delta)}. \tag{3.6.2}$$

Now, by (Problem 9 p. 194, [9]), (3.6.2) has a unique root. Hence $fuf^{-1} = u^{-1}$. Let $\phi: D_w \to C_{lmw} = \langle c; c^{lmw} \rangle$ such that $(uk^l)\phi = 1$and $k\phi = c$. Since $(fuf^{-1})\phi = c^{-l}$ and $(u^{-1})\phi = c^l$, we must have $c^{-l} = c^l$. Clearly this is not true for $w \geqslant 3$. It follows that $\beta - \varepsilon = \alpha - \delta$. Thus $fuf^{-1} = u$ unless $\beta - \varepsilon = \alpha - \delta = 0$. The former implies $[f, u] = 1$ in D_w. But the centralizer of $\langle u \rangle$ in D_w is $\langle u \rangle$. This implies $f \in \langle u \rangle$ contradicting $f \notin \langle u \rangle$. Hence $\beta - \varepsilon = \alpha - \delta = 0$. Thus $\alpha = \delta$ and $\beta = \varepsilon$.

COROLLARY 3.7. *Let $D = \langle u, k; (uk^l)^s \rangle$. Let $f, d \in D \backslash \langle u \rangle$ such that $fu^\alpha d = u^\beta$. If $fu^\delta d = u^\varepsilon$ then $\alpha = \delta$ and $\beta = \varepsilon$.*

Proof. Let ϕ be the obvious map of D onto a suitable D_w as in Lemma 3.6. Then it is clear $\alpha = \delta$ and $\beta = \varepsilon$.

Now $B = H *_{h^m = u} D$, where $D = \langle u, k; (uk^l)^s \rangle$. Let $N = D^B$. Then $N = (\prod *_{i=0}^{m-1} D_i)_{H_m}$, where $D_i = D^{h^i}$ and $H_m = \langle h^m \rangle = \langle u \rangle$. We shall also write N_w for the g.f.p. amalgamating H_m of $D_w(=(D_0)_w)$, $(D_1)_w,\dots,(D_{m-1})_w$. Using these notations we have

LEMMA 3.8. *Let $\omega, y \in N$ be such that $\omega = f_1 \cdots f_q$ and $\gamma = d_1 \cdots d_r$ are in normal form, the f_i and d_j coming from various $D_v \backslash H_m$. Let $q+r \geqslant 3$. Suppose that, for all α, $\omega u^\alpha \gamma^{-1} \notin K$. Then there exists a finite homomorphic image $\hat{N}$ of N in which, for all α, $\hat{\omega}\hat{u}^\alpha\hat{\gamma}^{-1} \notin \hat{K}$.*

Proof. N is Π_c (Stebe [13, Theorem 5]). Thus we can find a homomorphic image $\tilde{N} = (\tilde{D}_0 * \tilde{D}_1 * \cdots * \tilde{D}_{m-1})_{\tilde{H}_m}$ of N in which the $\tilde{D}_i$ are isomorphic and finite, and hence a homomorphic image N_w in which the $\bar{f}_i$ and $\bar{d}_j$ do not belong to $\bar{H}_m$. Thus if f_q and d_r belong to different D_v then $\|\bar{\omega}\bar{u}^\alpha\bar{\gamma}^{-1}\|$ has length $q+r-1$ in N_w. Hence, provided $q+r \geqslant 3$, $\bar{\omega}\bar{u}^\alpha\bar{\gamma}^{-1} \notin \bar{K}$ in N_w.

Suppose f_q and d_r lie in the same D_v and that $f_q u^\alpha d_r^{-1} \in H_m$ for some $\alpha = \alpha_0$. Then, by Corollary 3.7, α_0 is unique. Since $f_q, d_r \in D_v \backslash H_m$ and since D_v is Π_c we can find a finite homomorphic image of D_v and then a suitable $(D_v)_w$ in which $\bar{f}_q, \bar{d}_r \notin \bar{H}_m$. Thus, by Lemma 3.6 we can deduce that in N_w we have $\bar{f}_q\bar{u}^{\alpha_0}\bar{d}_r^{-1} \in \bar{H}_m$ but that $\bar{f}_q\bar{u}^\alpha\bar{d}_r^{-1} \notin \bar{H}_m$ for every other α. Thus, in N_w, $\|\bar{\omega}\bar{u}^\alpha\bar{\gamma}^{-1}\| = q+r-1$ except possibly in the case $\alpha = \alpha_0$. But $\omega u^{\alpha_0}\gamma^{-1} \notin K$ and since N is Π_c it is easy to find a finite homomorphic image of N and then a w_1 such that $\bar{\omega}\bar{u}^{\alpha_0}\bar{\gamma}^{-1} \notin \bar{K}$ in N_{w_1}. Clearly, then, for all α, $\bar{\omega}\bar{u}^\alpha\bar{\delta}^{-1} \notin \bar{K}$ in N_{ww_1}.

If, for all α, $f_q u^\alpha d_r^{-1} \notin H_m$, it is still possible that $\bar{f}_q\bar{u}^\alpha\bar{d}_r^{-1} \in \bar{H}_m$ in some $(D_v)_w$—but once again no other $f_q u^\beta d_r^{-1}$ can then map into $\bar{H}_m$ and we can proceed as above.

Thus in all cases where $q+r \geqslant 3$ we have shown that if $\omega u^\alpha \gamma^{-1} \notin K$ in N then there exists an N_w such that, for all α, $\bar{\omega}\bar{u}^\alpha\bar{\gamma}^{-1} \notin \bar{K}$ in N_w. That is, $\bar{\omega}^{-1}\bar{K}\bar{\gamma} \cap \langle \bar{u} \rangle = \varnothing$. But $\bar{K}$ is finite and N_w is Π_c (since D_w is). Thus there exists a finite homomorphic image $\hat{N}$ of N with the desired property.

LEMMA 3.9. *Let $\psi_w : N \to \tilde{N}_w = N/K_w$, where $K_w = \langle k^{lmw} \rangle^N$. Suppose $(fu^\alpha d)\psi_w$ and $(fu^\beta d)\psi_w \in \tilde{K}$, where $f, d \in D$. Then $\alpha = \beta$.*

Proof. Let $\tilde{f}\tilde{u}^\alpha\tilde{d} = \tilde{k}^i$ and $\tilde{f}\tilde{u}^\beta\tilde{d} = \tilde{k}^j$. Then $\tilde{f}\tilde{u}^{\alpha-\beta}\tilde{f}^{-1} = \tilde{k}^{i-j}$. Since $\tilde{D}_w = \langle \tilde{u}, \tilde{k}; (\tilde{u}\tilde{k}^l)^s = \tilde{k}^{lmw} = 1 \rangle$, $\tilde{u}$ has infinite order. But $\tilde{k}$ has finite order. Hence $\alpha = \beta$.

We are now ready to show that if $y, z \in B$ such that $z \notin EyF$, where E, F are H or K, then there exists an integer w' such that $\bar{z} \notin \bar{E}\bar{y}\bar{F}$ in $B_{w'}$.

LEMMA 3.10. *Let $y, z \in B$ such that $z \notin KyH$ (or HyK). Then there exists an integer w' such that $\bar{z} \notin \bar{K}\bar{y}\bar{H}$ (or $\bar{H}\bar{y}\bar{K}$) in $B_{w'}$.*

Proof. First, we note that $B = H *_{h^m = u} D$, where $D = \langle u, k; (uk^l)^s \rangle$. Let $N = D^B$. Then $N = (\prod *_{i=0}^{m-1} D_i)_{H_m}$, where $D_i = D^{h^i}$ and $H_m = \langle h^m \rangle = \langle u \rangle$. Clearly $B/N \cong \langle c; c^m \rangle$. Let $y = n_y h^i$ and $z = n_z h^j$, where $0 \leqslant i, j \leqslant m-1$ and $n_y, n_z \in N$. Thus $z \notin KyH$ if and only if $n_z \notin Kn_y H$ or equivalently $n_z h^v \notin Kn_y H_m$ for all $0 \leqslant v \leqslant m-1$. Let $\gamma = n_z h^v$. If $v = 1, \ldots, m-1$, then B/N is a finite homomorphic image of B such that $\gamma N \neq 1$ in B/N. This implies there exists an integer w_1 such that $\bar{\gamma} \notin \bar{K}\bar{n}_y \bar{H}_m$ in B_{w_1} for $v = 1, \ldots, m-1$ (indeed, $w = 1$ will certainly do). Suppose that for $v = 0$, there exists an integer w_2 such that $\bar{\gamma} \notin \bar{K}\bar{n}_y \bar{H}_m$ in B_{w_2}. Let $w' = w_1 w_2$. Then $\bar{\gamma} \notin \bar{K}\bar{n}_y \bar{H}_m$, $0 \leqslant v \leqslant m-1$ in $B_{w'}$. It follows that $\bar{z} \notin \bar{K}\bar{y}\bar{H}$ in $B_{w'}$. Hence we need only show that for $\gamma = n_z$ there exists an integer w_2 such that $\bar{\gamma} \notin \bar{K}\bar{n}_y \bar{H}_m$ in B_{w_2}, where $\gamma \in N$. To do so, we first note that $\gamma \notin Kn_y H_m$ if and only if $n_y u^\alpha \gamma^{-1} \notin K$ for all α. Let $\gamma = d_1 d_2 \cdots d_r$ and $n_y = f_1 f_2 \cdots f_q$ be reduced words in $N = (\prod *_{i=0}^{m-1} D_i)_{H_m}$.

Case 1. $q + r \geqslant 3$. Recall that we are given $\gamma = n_z \notin Kn_y H_m$, that is, $n_y u^\alpha \gamma^{-1} \notin K$ for all α. Then, by Lemma 3.8, there exists a finite homomorphic image $\hat{N}$ of N in which $\bar{n}_y \bar{u}^\alpha \bar{\gamma}^{-1} \notin \bar{K}$. Suppose $T = \ker N \to \hat{N}$. Then T is of finite index in N and hence in B. Thus there exists $S \subseteq T$ such that $|B : S| < \infty$ and $S \lhd B$. It follows that $\tilde{n}_y \tilde{u}^\alpha \tilde{\gamma}^{-1} \notin K$ in $\tilde{B} = B/S$. It then follows easily that there exists a w' such that $\bar{n}_y \bar{u}^\alpha \bar{\gamma}^{-1} \notin \bar{K}$ in $B_{w'}$. This suffices.

Case 2. $q = r = 1$. This means $n_y u^\alpha \gamma^{-1} = f_1 u^\alpha d_1^{-1}$. If f_1, d_1 belong to different factors of N then $\|f_1 u^\alpha d_1^{-1}\| = 2$ and we can proceed as before. Suppose, then, that f_1, d_1 belong to the same factor D_v of N. Then in N_w each $f_1 u^\alpha d_1^{-1}$ lies in $(D_v)_w$. Since $\bar{H}_m \cap \bar{K} = \langle \bar{1} \rangle$ in N_w we see, at least when $v \neq 0$, that no $\bar{n}_y \bar{u}^\alpha \bar{\gamma}^{-1}$ lies in $\bar{K}$ in N_w.

So assume $f_1, d_1 \in D$, the free factor which contains K and consider the map $\psi_w : N \to \tilde{N}_w = N/K_w$. By Lemma 3.9 there exists at most one α (say $\alpha = \alpha_0$) such that $(f_1 u^{\alpha_0} d_1^{-1}) \psi_w \in \tilde{K}$. Then, for all multiples w_1 of w, $f_1 u^{\alpha_0} d_1^{-1}$ remains the only element of that type which might be mapped into $\tilde{K}$ in $\tilde{N}_{w_1}$. Since N is Π_c, there exists a finite image of N such that the image of $f_1 u^{\alpha_0} d_1^{-1}$ is not in the image of K. One then eaily finds an N_{w_2} such that, for all α, $\bar{f}_1 \bar{u}^\alpha \bar{d}_1^{-1} \notin \bar{K}$ in N_{w_2}, and one finishes the argument by finding a $B_{w'}$ as in Case 1.

Case 3. $q = 0$ or $r = 0$. Without loss of generality, let $r = 0$. Thus $\gamma \in H_m$. Suppose $\|n_y\| = 2$ in N. Then there exists an integer w such that $\|\bar{n}_y\| = \|n_y\|$ in N_w. But this implies $\bar{n}_y \bar{u}^\alpha \bar{\gamma}^{-1} \notin \bar{K}$ for each u^α, by a length argument. It follows that $\bar{z} \notin \bar{K}\bar{y}\bar{H}$ in some suitable B_w.

Thus we may assume $\|n_y\| = 1$ in N. Let θ be any homomorphism from N. Since $\gamma \in H_m$ we see that $(n_y u^\alpha \gamma^{-1})\theta \notin K\theta$ if and only if for all $i, j \in \mathbb{Z}$,

$n_y\theta \neq (k\theta)^i(u\theta)^j$. Thus if $n_y \notin D$ it is easy to find an integer w such that $\bar{n}_y \notin D_w$ in N_w whence $\bar{n}_y\bar{u}^\alpha\bar{\gamma}^{-1} \notin \bar{K}$ in N_w. Thus, again, $\bar{z} \notin \bar{K}\bar{y}\bar{H}$ in some B_w.

Consequently we assume $n_y \in D$, where

$$D = \langle u, k; (uk^l)^s \rangle = \langle X; X^s \rangle * K$$

and where $X = uk^l$. Now n_y is not of the form $k^iu^j = k^i(Xk^{-l})^j$ since $\gamma \notin Kn_yH_m$. Mapping D onto $\hat{D}_w = \langle \hat{X}; \hat{X}^s \rangle * \langle \hat{k}: \hat{k}^{lmw} \rangle$, where $w \geqslant 3$, we see that $\hat{n}_y$ is not of the form $\hat{k}^i(\hat{X}\hat{k}^{-l})^j = \hat{k}^i\hat{u}^j$. Thus $\hat{n}_y\hat{u}^\alpha\hat{\gamma}^{-1} \notin K$. This suffices, as usual. This completes the proof of the lemma.

Lemma 3.11. *Let $y, z \in B$ such that $z \notin HyH$ (or KyK). Then there exists an integer w' such that $\bar{z} \notin \bar{H}\bar{y}\bar{H}$ (or $\bar{K}\bar{y}\bar{K}$) in $B_{w'}$.*

Proof. As in Lemma 3.10, let $N = D^B$, where $D = \langle u, k; (uk^l)^s \rangle$ and $u = h^m$. Let $y = n_yh^i$ and $z = n_zh^j$, where $n_y, n_z \in N$ and $0 \leqslant i, j \leqslant m-1$. Then $z \notin HyH$ if and only if $n_z \notin Hn_yH$ or, equivalently, none of the m^2 elements $h^\alpha n_z h^\beta \in H_m n_y H_m$, where $H_m = \langle h^m \rangle = \langle u \rangle$, and $0 \leqslant \alpha, \beta \leqslant m-1$. Let $\gamma = h^\alpha n_z h^\beta$. As in Lemma 3.10, for each $\gamma \notin N$ we have $\bar{\gamma} \notin \bar{H}_m\bar{n}_y\bar{H}_m$ in B_1. Thus if we can show that for each $\gamma \in N$ there exists an integer w^* such that $\bar{\gamma} \notin \bar{H}_m\bar{n}_y\bar{H}_m$ in B_{w^*}, then, by letting w' be the product of all these integers, $\bar{z} \notin \bar{H}\bar{y}\bar{H}$ in $B_{w'}$. Thus we need only consider the case of $\gamma \in N$ with $\gamma \notin H_m n_y H_m$. Let $\gamma = d_1 \cdots d_r$ and $n_y = f_1 \cdots f_q$, where d_i, f_j are elements of the various $D_v \backslash H_m$ with $D_v = D^{h^v}$ and $0 \leqslant v \leqslant m-1$. Since each D_v is ${}_{\mathrm{R}}\mathrm{F}$ and H_m-separable, there exists $M \lhd N$ such that $|D_v : M \cap D|$ is finite for all v and d_iM and f_iM are elements of the appropriate $\hat{D}_v \backslash \hat{H}_v = (D_vM/M) \backslash (H_mM/M)$. Therefore, in $\hat{N} = N/M$, $l(\hat{\gamma}) = l(\gamma)$ and $l(\hat{n}_y) = l(n_y)$. Thus if $r \neq q$ then $\hat{\gamma} \notin \hat{H}_m\hat{n}_y\hat{H}_m$. Since $|\hat{H}_m\hat{n}_y\hat{H}_m|$ is finite and $\hat{N} \in {}_{\mathrm{R}}\mathrm{F}$, there exists $L \lhd_f N$ such that $\hat{\gamma}\hat{L}$ is not in the image of $\hat{H}_m\hat{n}_y\hat{H}_m$ in $\hat{N}/\hat{L}$. But this implies that there exists an integer w such that $\bar{z} \notin \bar{H}\bar{y}\bar{H}$ in B_w. Thus we need only consider the case when $r = q$. Moreover, if $r = q = 0$ then $\gamma, n_y \in H_m$, which implies $y, z \in H$ contradicting $z \notin HyH$. Hence we need only consider $r = q \geqslant 1$.

First, we note that $\gamma \in H_m n_y H_m$ if and only if there exist integers $i_0, i_1, \ldots, i_r$ such that

$$\begin{aligned} d_1 &= u^{-i_0}f_1u^{i_1}, \\ d_2 &= u^{-i_1}f_2u^{i_2}, \\ &\vdots \\ d_r &= u^{-i_{r-1}}f_ru^{i_r}. \end{aligned} \tag{3.11.1}$$

Hence we shall say that a single equation $d_\lambda = X_\lambda f_\lambda Y_\lambda$ is *acceptable* in the appropriate D_v if there exist integers $\alpha_\lambda, \beta_\lambda$ such that $d_\lambda = u^{-\alpha_\lambda} f_\lambda u^{\beta_\lambda}$. Further, we say that the system of equations

$$d_\lambda = X_\lambda f_\lambda Y_\lambda \qquad (\lambda = 1, 2, \ldots, r) \tag{3.11.2}$$

is *compatible* if each equation is acceptable and there exist integers α_λ, β_λ such that $\beta_\lambda = \alpha_{\lambda+1}$ for $\lambda = 1, \ldots, r-1$.

Case 1. There exists an integer λ such that $d_\lambda = X_\lambda f_\lambda Y_\lambda$ is not acceptable. Then, by Lemma 3.2 (Tang [16]), there exists a finite homomorphic image $\hat{D}_v$ of D_v such that $\hat{d}_\lambda \notin \hat{H}_m \hat{f}_\lambda \hat{H}_m$. Hence there exists an integer w^* such that $\bar{d}_\lambda \notin \bar{H}_m \bar{f}_\lambda \bar{H}_m$ in B_{w^*}, whence $\bar{z} \notin \bar{H}\bar{y}\bar{H}$ in B_{w^*}.

Case 2. Every equation in (3.11.2) is acceptable but (3.11.2) is not compatible. By Corollary 3.7, the solution $X_\lambda = u^{-\alpha_\lambda}$, $Y_\lambda = u^{\beta_\lambda}$ is unique to each pair d_λ, f_λ. Since (3.11.2) is not compatible there exists an integer μ such that $\beta_\mu \neq \alpha_{\mu+1}$. Now, by Lemma 3.6, for $w \geqslant 3$, $\bar{d}_\lambda = \bar{X}_\lambda \bar{f}_\lambda \bar{Y}_\lambda$ has a unique solution in the appropriate free factor $(D_v)_w$ of N_w. This implies that there exists an integer w such that if $y \notin H_m n_y H_m$ in N then $\bar{y} \notin \bar{H}_m \bar{n}_y \bar{H}_m$ in N_w. We now rechoose w, if necessary, so that, in addition, $\bar{f}_\lambda \notin \bar{H}_m$ for each λ $(\lambda = 1, 2, \ldots, r)$. For each $(D_v)_w$ (briefly D_w) we now consider the natural homomorphism $\theta\colon D_w \to \bar{\bar{E}}_w$ where $\bar{\bar{E}}_w = \langle \bar{\bar{u}}, \bar{\bar{k}};\, (\bar{\bar{u}}\bar{\bar{k}}^l)^s = \bar{\bar{k}}^{lmw} = \bar{\bar{u}}^e = 1 \rangle$ with $e > \max(2\,|\alpha_\lambda|, 2\,|\beta_\lambda|)$ and such that $\bar{\bar{f}}_\lambda \notin \langle \bar{\bar{u}} \rangle$. (This is possible since $\bar{f}_\lambda \notin \langle \bar{u} \rangle$ and D_w is Π_c.) Suppose $\bar{\bar{d}}_\lambda = \bar{\bar{u}}^{\delta_\lambda} \bar{\bar{f}}_\lambda \bar{\bar{u}}^{\varepsilon_\lambda}$ is another solution to $\bar{\bar{d}}_\lambda = \bar{\bar{X}}_\lambda \bar{\bar{f}}_\lambda \bar{\bar{Y}}_\lambda$ in $\bar{\bar{E}}_w$. Then

$$\bar{\bar{f}}_\lambda^{-1} 1_{\bar{\bar{u}}}^{\alpha_\lambda + \delta_\lambda} \bar{\bar{f}}_\lambda = \bar{\bar{u}}^{\beta_\lambda - \varepsilon_\lambda}. \tag{3.11.3}$$

Now $\bar{\bar{E}}_w = \langle \bar{\bar{u}}, \bar{\bar{v}};\, (\bar{\bar{u}}\bar{\bar{v}})^s = \bar{\bar{v}}^{mw} = \bar{\bar{u}}^e = 1 \rangle *_{\bar{\bar{v}} = \bar{\bar{k}}^l} \langle \bar{\bar{k}};\, \bar{\bar{k}}^{lmw} \rangle = Q *_{\bar{\bar{v}} = \bar{\bar{k}}^l} K$. $\bar{\bar{u}}^{\alpha_\lambda + \delta_\lambda}$ and $\bar{\bar{u}}^{\beta_\lambda - \varepsilon_\lambda}$ are either both 1 or both are of minimal length in their conjugacy classes in $\bar{\bar{E}}_w$. (No nontrivial power of $\bar{\bar{u}}$ can be conjugate to any power of $\bar{\bar{v}}$ by Corollary 2, [7]). In this latter case, $\bar{\bar{u}}^{\alpha_\lambda + \delta_\lambda}$ and $\bar{\bar{u}}^{\beta_\lambda - \varepsilon_\lambda}$ are conjugate in Q; indeed using Corollary 2 of [7] we see that (3.11.3) implies $\bar{\bar{f}}_\lambda$ must be in Q.

Thus we have either

(i) $\alpha_\lambda + \delta_\lambda \not\equiv 0 \bmod e$ and $\bar{\bar{f}}_\lambda \in Q$ or

(ii) $\alpha_\lambda + \delta_\lambda \equiv \beta_\lambda - \varepsilon_\lambda \equiv 0 \bmod e$.

In case (i), Corollary 2, [7] implies $\bar{\bar{f}}_\lambda \in \langle \bar{\bar{u}} \rangle$ contrary to the choice of e. Thus case (ii) holds and we deduce that for each λ the equation $\bar{\bar{d}}_\lambda = \bar{\bar{X}}_\lambda \bar{\bar{f}}_\lambda \bar{\bar{Y}}_\lambda$ has the unique solution $\bar{\bar{X}}_\lambda = \bar{\bar{u}}^{-\alpha_\lambda}$, $\bar{\bar{Y}}_\lambda = \bar{\bar{u}}^{\beta_\lambda} \bmod e$. In particular, $\bar{\bar{d}}_\mu = \bar{\bar{X}}_\mu \bar{\bar{f}}_\mu \bar{\bar{Y}}_\mu$ and $\bar{\bar{d}}_{\mu+1} = \bar{\bar{X}}_{\mu+1} \bar{\bar{f}}_{\mu+1} \bar{\bar{Y}}_{\mu+1}$ have in $\bar{\bar{N}}_w$, the g.f.p. of the $(\bar{\bar{E}}_v)_w$ amalgamating $\langle \bar{\bar{u}} \rangle$ of order e, unique solutions $\bar{\bar{u}}^{-\alpha_\mu}$, $\bar{\bar{u}}^{\beta_\mu}$; $\bar{\bar{u}}^{-\alpha_{\mu+1}}$, $\bar{\bar{u}}^{\beta_{\mu+1}}$. Thus in $\bar{\bar{N}}_w$ the equations $\bar{\bar{d}}_\lambda = \bar{\bar{X}}_\lambda \bar{\bar{f}}_\lambda \bar{\bar{Y}}_\lambda$ $(\lambda = 1, 2, \ldots, r)$ are acceptable but, because $\beta_\mu \neq a_{\mu+1}$ (mod e), they are not compatible.

Thus $\bar{\bar{y}} \notin \bar{\bar{H}}_m \bar{\bar{n}}_y \bar{\bar{H}}_m$ in $\bar{\bar{N}}_w$. Now, since $B_w = \langle h; h^{lmw} \rangle *_{h^m = u} \langle u, k; (uk^l)^s = k^{lmw} = u^{lw} = 1 \rangle$. It is clear we can choose w', sufficiently large so that $\hat{y} \notin \hat{H}_m \hat{n}_y \hat{H}_m$ and hence $\hat{z} \notin \hat{H}\hat{y}\hat{H}$ in $B_{w'}$.

This completes the proof of the lemma.

To prove our main result we shall need to show that most polyhedral groups are c.s. The question of whether or not all polyhedral groups are c.s. remains open.

LEMMA 3.12. *Let* $P = \langle a, b; (ab)^n = a^{ln} = b^{mn} = 1 \rangle$. *Then P is c.s.*

Proof. Since every polyhedral group is a degenerate Fuchsian group, it follows from Theorem 3.10 (Stebe [15]), that every element of infinite order in P is c.d. Thus to show that P is c.s. we need only show that every element of finite order in P is c.d. Let $u, v \in P$ be of finite order and let $u \nsim_P v$. Since polyhedral groups are $_R$F [3] the case where either $u = 1$ or $v = 1$ is easily dealt with. Hence we may assume $u \neq 1$ and $v \neq 1$. Since every element of finite order is conjugate to an element of $\langle a \rangle \cup \langle b \rangle \cup \langle ab \rangle$, we can w.l.o.g. assume u, v to be elements of $\langle a \rangle$, $\langle b \rangle$ or $\langle ab \rangle$. Thus we need only consider the following three cases:

Case 1. $u = a^r$ and $v = b^s$. Suppose $\langle v \rangle = \langle b^\alpha \rangle$, where $\alpha \mid mn$. If $\alpha \neq 1$ then $\langle v \rangle$ is a proper subgroup of $\langle b \rangle$. Let $P_1 = \langle x, y; (xy)^n = x^{ln} = y^\alpha = 1 \rangle$. Let $\theta: P \to P_1$ be a homomorphism of P onto P_1 such that $a\theta = x$ and $b\theta = y$. Since $\alpha \neq 1$, P_1 is a polyhedral group. Thus P_1 is $_R$F. Hence there exists $N \lhd_f P_1$ such that $\bar{x} = xN$ has order ln in $\bar{P}_1 = P_1/N$. Let ϕ be the canonical homomorphism of P_1 onto $\bar{P}_1$. Then

$$u\theta\phi = a^r\theta\phi = x^r\phi = \bar{x}^r \neq 1,$$

$$v\theta\phi = b^s\theta\phi = y^s\phi = 1\phi = 1.$$

Clearly $\bar{x}^r \nsim_{\bar{P}_1} 1$. Hence there exists a finite image of P in which the images of u and v are not conjugate. Thus to complete the proof of this case we can assume that $\langle u \rangle = \langle a \rangle$ and $\langle v \rangle = \langle b \rangle$. If $ln \neq mn$ then we can easily find a finite image of P in which the images of a and b have orders ln and mn, respectively, whence the images of u and v cannot be conjugates. Hence we can assume $ln = mn$, that is, $m = l$. This implies that $P = \langle a, b; (ab)^n = a^{ln} = b^{ln} = 1 \rangle$. Let $\psi: P \to C_n = \langle c; c^n \rangle$ such that $a\psi = c$ and $b\psi = 1$. Since $a\psi \neq 1$ it follows that $u\psi \neq 1$. On the other hand, $b\psi = 1$ implies that $v\psi = 1$, whence $u\psi \nsim_{C_n} v\psi$. This shows that u, v are c.d.

Because of the lack of symmetry in the presentation of P we must also consider:

Case 2. $u = a^r$ and $v = (ab)^s$. If either $\langle u \rangle \subsetneq \langle a \rangle$ or $\langle v \rangle \subsetneq \langle ab \rangle$ then we can apply a similar argument as in Case 1 to show that there exists a finite image of P in which the images of u and v are not conjugates. So we can assume $\langle u\rangle = \langle a\rangle$ and $\langle v\rangle = \langle ab\rangle$. If $ln \neq n$ then the argument is again the same as Case 1. If $ln = n$, then $l = 1$. Thus $P = \langle a, b; (ab)^n = a^n = b^{mn} = 1\rangle$. Let $\psi: P \to C_n = \langle c; c^n\rangle$ such that $a\psi = c$ and $b\psi = c^{-1}$. Then $a\psi \neq 1$ implies $u\psi \neq 1$. On the other hand $(ab)\psi = 1$, whence $v\psi = 1$. It follows that u, v are c.d.

Case 3. $u = a^r$ and $v = a^s$. Clearly $r \neq s$. Moreover, if $n = 1$, P is cyclic whence trivially c.s. Thus we can assume $n \neq 1$. Let p be a prime dividing n and let $C_p = \langle c; c^p\rangle$. Let $\psi: P \to C_p$ such that $a\psi = 1$, $b\psi = c$. Then $\langle a\rangle \subset \ker \psi = N$. By Theorem 1.5 (Sah [12]), N is a group of genus 0 and signature

$$\Big\{\underbrace{e(1),\dots, e(1)}_{p \text{ times}}, \frac{e(2)}{p}, \frac{e(3)}{p}\Big\},$$

where $e(1) = ln$, $e(2) = mn$ and $e(3) = n$. Thus,

$$N = \langle x_1,\dots, x_p, x_{p+1}, x_{p+2}; x_i^{ln}, (1 \leqslant i \leqslant p),$$
$$x_{p+1}^{mn/p}, x_{p+2}^{n/p}, x_1,\dots, x_p x_{p+1} x_{p+2}\rangle.$$

Moreover, if a is not an x_i $(1 \leqslant i \leqslant p)$, a is conjugate to one of these elements. So, for convenience, we can assume a to be x_1 (say). Now P is an extension of N by a p-cycle. Thus if $P/N = \langle yN; (yN)^p\rangle$ then $\langle x_1\rangle,\dots, \langle x_p\rangle$ are the p conjugates of $\langle x_1\rangle$ under y^i, $0 \leqslant i \leqslant p-1$. Let $\phi: N \to Z = \langle z; z^{ln}\rangle$ such that $x_1\phi = z$, $x_2\phi = z^{-1}$ and $x_j\phi = 1$ for $3 \leqslant j \leqslant p+2$. Let $M = \ker \phi$. Since $r \neq s$, $x_1^r\phi \neq x_1^s\phi$. Now, we have chosen $u = a^r = x_1^r$ and $v = a^s = x_1^s$, whence $u\phi \not\sim_Z v\phi$. Moreover $[P : M]$ is finite. Therefore there exists $L \lhd_f P$ with $L \subseteq M$. Since $u\phi \not\sim_Z v\phi$, it follows that $uL \not\sim_{N/L} vL$. Now $a^{y^i} = x_1^{y^i} = x_i$. Thus $u \not\sim_{y^i} v$ implies that $uL \not\sim_{N/L} vL$. Together with the above argument, it follows that if $u \not\sim_P v$ then $uL \not\sim_{P/L} vL$. This implies that u, v are c.d.

This proves the lemma.

LEMMA 3.13. *B_{ws} is c.s. for all $w \geqslant 1$.*

Proof. Consider $B_w = H' *_{h^m = u} C$ where $H' = \langle h; h^{lmws}\rangle$ and $C = \langle u, k; (uk^l)^s = u^{lws} = k^{lmws} = 1\rangle$. Now $C = D *_{v = k^l} K'$ where $D = \langle u, v; (uv)^s = u^{lws} = v^{mws} = 1\rangle$ and $K' = \langle k; k^{lmws}\rangle$. By Lemma 3.12, D is c.s. for $w > 1$. Since K' and H' are finite cycles, by repeated applications of theorem A, B_{ws} is c.s.

4. Main Result

To show that G is c.s. we make use of the fact that G is an *HNN* extension of B. If $x, y \in B$ then, by Collins' lemma (Theorem 3(i), [6]), $x \sim_G y$ only if there exists a finite sequence of elements $z_1, z_2, ..., z_n \in H \cup K$ such that

$$x \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_n \sim_B y. \tag{4.1}$$

We shall call (4.1) *a conjugating sequence of length* $n+2$ for x and y.

Lemma 4.1. *Let $x, y \in B \subset G$. If $x \sim_G y$, then there exists a conjugating sequence of length at most 4 of the form*:

$$x \sim_B z_1 \sim_{t^*} z_2 \sim_B y, \tag{4.2}$$

where $z_1, z_2 \in H \cup K$.

Proof. First, we note that in (4.1) any successive conjugations by elements of B can be reduced to a conjugation by a single element of B. Since $H \cap K = \langle 1 \rangle$, (4.1) cannot contain two successive conjugations by t nor two successive conjugations by t^{-1}. Also any successive conjugations by t^{ε} and $t^{-\varepsilon}$, where $\varepsilon = \pm 1$, can be deleted. Hence in (4.1) conjugations by elements of B and $\{t, t^{-1}\}$ can be assumed to alternate. Let $z_i, z_{i+1} \in H \cup K$. Suppose $z_i \sim_B z_{i+1}$. Then $z_i = z_{i+1}$ by Lemma 3.4. Hence this step can be deleted. It follows that (4.1) can be reduced to (4.2).

Since G_w is likewise, an *HNN* extension by $\langle t \rangle$ of B_w we get, by using Lemma 3.1.

Lemma 4.2. *Let $\bar{x}, \bar{y} \in B_w \subset G_w$. If $\bar{x} \sim_{G_w} \bar{y}$ then there exists a conjugating sequence of length at most 4 of the form*

$$\bar{x} \sim_{B_w} \bar{z}_1 \sim_{\bar{t}^*} \bar{z}_2 \sim_{B_w} \bar{y},$$

where $\bar{z}_1, \bar{z}_2 \in \bar{H} \cup \bar{K}$.

Now let $x, y \in G \setminus B$ be given by (2.1) and suppose $\mu = \lambda \geqslant 1$. Consider the following set of equations where the u_i and $v_i \in B$ and the equations are to be solved for W_i and Z_i in $H \cup K$,

$$\begin{aligned} u_1 &= W_1^{-1} v_1 Z_1, \\ u_2 &= W_2^{-1} v_2 Z_2, \\ &\vdots \\ \mu_\lambda &= W_\lambda^{-1} v_\lambda Z_\lambda. \end{aligned} \tag{4.3}$$

A pair of elements ρ_i, σ_i of B will be called an *admissible* solution of the ith equation of (4.3) if and only if $u_i=\rho_i^{-1}v_i\sigma_i$, where $\rho_i\in H$ or K according as $e(i)=-1$ or $e(i)=1$, respectively, and $\sigma_i\in H$ or K according as $e(i+1)=1$ or $e(i+1)=-1$, respectively. (Note: the pair $\rho_\lambda, \sigma_\lambda$ will be called admissible if and only if $u_\lambda=\rho_\lambda^{-1}v_\lambda\sigma_\lambda$, where $\rho_\lambda\in H$ or K according as $e(\lambda)=-1$ or 1 and $\sigma_\lambda\in H$ or K according as $e(1)=1$ or $e(1)=-1$, respectively.) A set of admissible solutions $\rho_1, \sigma_1, \ldots, \rho_\lambda, \sigma_\lambda$ to (4.3) is said to be *complete* if for each i, $1\leqslant i\leqslant\lambda$, $t^{-e(i)}\sigma_{i-1}t^{e(i)}=\rho_i$, where $\sigma_0=\sigma_\lambda$.

Now, by Collins' lemma (Dyer [6, p. 39]), for $x, y\in G\backslash B$ with $\|x\|=\|y\|\geqslant 1$, we have $x\sim_G y$ if and only if $y\sim_{H\cup K}x^*$, where x^* is a cyclic permutation of x. Assuming for the moment that $x^*=x$ we see, by (iv), (v), (vi) of Dyer [6, p. 39], that $y\sim_{H\cup K}x$ if and only if $e(i)=f(i)$ (see 2.1) and there exists a complete set of solutions to Eqs. (4.3).

We need the following lemmas.

LEMMA 4.3. (i) *Let $x, y\in B$ such that $x=h^iyk^j$. If $x=h^\alpha yk^\beta$ then $i=\alpha$ and $j=\beta$.*

(ii) *Let $x, y\in B$ such that not both $x, y\in H$ (or K). If $x=h^iyh^j$ (or k^iyk^j) and $x=h^\alpha yh^\beta$ (or $k^\alpha yk^\beta$) then $i=\alpha$ and $j=\beta$.*

Proof. (i) $h^iyk^j=h^\alpha yk^\beta$ implies $y^{-1}h^{i-\alpha}y=k^{\beta-j}$. Hence, by Lemma 3.4, $i-\alpha=\beta-j=0$.

(ii) $h^iyh^j=h^\alpha yh^\beta$ implies $y^{-1}h^{i-\alpha}y=h^{\beta-j}$. Hence, by Lemma 3.4, either $i-\alpha=\beta-j=0$ (as required) or $i-\alpha=\beta-j=\gamma\neq 0$. But the centralizer of h^γ in the 1-relator group B is H, by Newman [11]. Thus $y\in H$ which is a contradiction.

LEMMA 4.4. (i) *Let $x, y\in B$ such that $x=h^iyk^j$. Then, for all integers w if $\bar{x}=\bar{h}^\alpha\bar{y}\bar{k}^\beta$ in B_w then $\bar{h}^i=\bar{h}^\alpha$ and $\bar{k}^j=\bar{k}^\beta$.*

(ii) *Let $x, y\in B$ such that not both $x, y\in H$ (or K) and such that $x=h^iyh^j(k^iyk^j)$. Then, for all suitably large w if $\bar{x}=\bar{h}^\alpha\bar{y}\bar{h}^\beta(\bar{k}^\alpha\bar{y}\bar{k}^\beta)$ in B_w then $\bar{h}^i=\bar{h}^\alpha$ and $\bar{h}^j=\bar{h}^\beta$ ($\bar{k}^i=\bar{k}^\alpha$ and $\bar{k}^j=\bar{k}^\beta$).*

Proof. (i) Suppose $\bar{x}=\bar{h}^i\bar{y}\bar{k}^j=\bar{h}^\alpha\bar{y}\bar{k}^\beta$ in B_w. Then $\bar{h}^{i-\alpha}\sim_{B_w}\bar{k}^{\beta-j}$. Hence $i-\alpha\equiv\beta-j\equiv 0 \pmod{lmw}$, by Lemma 3.1.

(ii) Since B is Π_c (Lemma 2.2) we can choose w such that $\bar{y}\notin\bar{H}$.

As in (i) $\bar{x}=\bar{h}^\alpha\bar{y}\bar{h}^\beta$ implies $\bar{y}^{-1}\bar{h}^{\alpha-i}\bar{y}=\bar{h}^{j-\beta}$. By Lemma 3.1, it follows that $\alpha-i\equiv j-\beta \bmod lmw$. If $\alpha-i\equiv 0 \bmod lmw$ then the lemma is proved. Thus we can assume $\alpha-i\not\equiv 0 \bmod lmw$. Now $[\bar{y}, \bar{h}^{\alpha-i}]=1$. Also $B_w=\bar{K}*_{\bar{k}^l=\bar{v}}\bar{D}_w$, where $\bar{K}=\langle k; k^{lmw}\rangle$, and $\bar{\bar{D}}_w=\langle\bar{h}, \bar{v}; (\bar{h}^m\bar{v})^s=\bar{h}^{lmw}=\bar{v}^{mw}=1\rangle$. Suppose $\|\bar{y}\|>1$ in B_w. Let $\bar{y}=d_1k_1\cdots d_nk_n$ (say) where

$d_i \in \bar{\bar{D}}_w \backslash \langle \bar{v} \rangle$ and $k_i \in \bar{K} \backslash \langle \bar{v} \rangle$. (Since $[\bar{y}, \bar{h}^{\alpha-i}]=1$, it follows that init $\bar{y} \notin \bar{K}$.) Consider

$$k_n^{-1} d_n^{-1} \cdots k_1^{-1} d_1^{-1} \bar{h}^{\alpha-i} d_1 k_1 \cdots d_n k_n = \bar{h}^{\alpha-i}. \tag{4.5}$$

Now $\|\bar{y}\|>1$ implies that for (4.5) to hold $d_1^{-1}\bar{h}^{\alpha-i}d_1 \in \langle \bar{v} \rangle$. Now either $\bar{h}^{\alpha-i}=1$, whence $\bar{h}^\alpha = \bar{h}^i$, or $\bar{h}^{\alpha-i} \neq 1$. In the latter case by applying a similar argument to the proof of Lemma 3.1(i), we can show $d_1^{-1}\bar{h}^{\alpha-i}d_1 \notin \langle \bar{v} \rangle$. Hence $\|\bar{y}\| \leqslant 1$ in B_w. This implies $\bar{y} \in \bar{\bar{D}}_w$. Now $\bar{\bar{D}}_w = \bar{E} *_{\bar{u}=\bar{h}^m} \bar{H}$, where $\bar{H} = \langle \bar{h}; \bar{h}^{lmw} \rangle$ and $\bar{E} = \langle \bar{u}, \bar{v}; (\bar{u}\bar{v})^s = \bar{u}^{lw} = \bar{v}^{mw} = 1 \rangle$. By the choice of w, $\bar{y} \notin \bar{H}$, and yet $[\bar{y}, h^{\alpha-i}]=1$. This implies that either $\bar{h}^{\alpha-i}=1$, in which case $\bar{h}^\alpha = \bar{h}^i$ whence $\bar{h}^j = \bar{h}^\beta$ or $\bar{h}^{\alpha-i} \in \langle \bar{u} \rangle$ and $\bar{y} \in \bar{E}$. (To see this assume $\|\bar{y}\|>1$ and use Corollary 2 of [7].) In the latter case, we have $\bar{y}^{-1}\bar{u}^\delta \bar{y} = \bar{u}^\delta$ for some integer δ. Since the centralizer of $\bar{u}^\delta$ in $\bar{E}$ is $\langle \bar{u} \rangle$ (Hoare, Karrass, and Solitar [7, Corollary 2]), it follows that $\bar{y} \in \langle \bar{u} \rangle \subseteq \bar{H}$, contradicting $\bar{y} \notin \bar{H}$. This completes the proof.

We are now ready to prove our main theorem.

THEOREM 4.5. *The group* $G = \langle b, t; (t^{-1}b^l t b^m)^s \rangle$, $s>1$, *is c.s.*

Proof. G is an HNN extension of $B' = \langle b_0, b_1; (b_1^l b_0^m)^s \rangle$ by $\langle t \rangle$, where $b_i = t^{-i} b t^i$, $i = 0, \pm 1, \ldots$. For convenience, we shall write $k = b_1$ and $h = b_0$. Then $B = B' = \langle h, k; (h^m k^l)^s \rangle$. Let $x, y \in G$ such that $x \not\sim_G y$. Without loss of generality, we can assume x, y to be cyclically reduced and of minimal length in their conjugacy classes. Regarding the cyclically reduced forms (2.1), for suitable choices of w, the lengths of $\bar{x}$ and $\bar{y}$ in G_w remain the same as those of x and y in G respectively. Thus, by Collins' results (Theorems 3 and 13 of [6]) we need only consider the cases when $\|x\| = \|y\|$ (see also Dyer [6, p. 45] or Tang [16, pp. 382–383]).

Case 1. $\|x\| = \|y\| = 0$. Suppose $\bar{x} \sim_{G_w} \bar{y}$ for some integer w. Then by Lemma 4.2, we must have one of the following cases:

(i) $\bar{x} \sim_{B_w} \bar{y}$,

(ii) $\bar{x} \sim_{\bar{t}*} \bar{y}$,

(iii) $\bar{x} \sim_{B_w} \bar{z}_1 \sim_{\bar{t}*} \bar{y}$,

(iv) $\bar{x} \sim_{\bar{t}*} \bar{z}_2 \sim_{B_w} \bar{y}$,

(v) $\bar{x} \sim_{B_w} \bar{z}_1 \sim_{\bar{t}*} \bar{z}_2 \sim_{B_w} \bar{y}$,

where $\bar{z}_1, \bar{z}_2 \in \bar{H} \cup \bar{K}$. Now $x \not\sim_G y$ implies $x \not\sim_B y$ and $x \not\sim_{t*} y$. By Theorem C, $x \not\sim_B y$ implies that there exists an integer w_1 such that

$\bar{x} \not\sim_{B_{w_1}} \bar{y}$. Also, by Lemma 3.2, $x \not\sim_{t^*} y$ implies that there exists an integer w_2 such that $\bar{x} \not\sim_{t^*} \bar{y}$ in G_{w_2}. Let $w = w_1 w_2$. Then cases (i) and (ii) do not apply. (Throughout the subsequent argument we shall vary w to a larger value to suit out purpose.) Let a_1, a_2 denote either h or k as appropriate.

(A) Suppose $x \sim_B a_1^i$ and $a_2^j \sim_B y$. Since $x \not\sim_G y$, we must have $a_1^i \not\sim_{t^*} a_2^j$, $a_1^i \not\sim_{t^*} y$, and $x \not\sim_{t^*} a_2^j$. Now, by Lemma 3.4, a_1^i and a_2^j are uniquely determined by x and y, respectively. Choose w to be larger than $2(\max\{|i|, |j|\})$. Then, by Lemmas 3.1 and 3.2, we have, in G_w, $\bar{x} \sim_{B_w} \bar{a}_1^i$, $\bar{a}_2^j \sim_{B_w} \bar{y}$, where $\bar{a}_1^i$ and $\bar{a}_2^j$ are uniquely determined, and that $\bar{a}_1^i \not\sim_{\bar{t}^*} \bar{a}_2^j$, $\bar{a}_1^i \not\sim_{\bar{t}^*} \bar{y}$, and $\bar{x} \not\sim_{\bar{t}^*} \bar{a}_2^j$. Thus, for some sufficiently larger integer w, (iii), (iv), and (v) all fail in G_w, whence $\bar{x} \not\sim_{G_w} \bar{y}$ for some G_w. Hence, by Theorem B, x, y are c.d.

(B) Suppose $x \sim_B a_1^i$ and $y^B \cap (H \cup K) = \varnothing$. Clearly $a_1^i \not\sim_{t^*} y$ (since $x \not\sim_G y$). By Lemma 3.3, there exists an integer w sufficiently large so that $\bar{y}^{B_w} \cap (\bar{H} \cup \bar{K}) = \varnothing$. Now, by Lemma 3.1, $\bar{x} \sim_{B_w} \bar{a}_1^i$ with a_1^i uniquely determined by $\bar{x}$. Applying Lemma 3.2, we can further assume w large enough so that $\bar{a}_1^i \not\sim_{\bar{t}^*} \bar{y}$ in G_w. Thus (iii), (iv), and (v) cannot hold in some G_w. Hence, as in (A), x, y are c.d.

(C) Suppose $x^B \cap (H \cup K) = \varnothing$ and $a_2^j \sim_B y$. Then, as in (B), we can show x, y are c.d.

(D) Suppose $x^B \cap (H \cup K) = y^B \cap (H \cup K) = \varnothing$. Then, by Lemma 3.3, we can find an integer w sufficiently large so that $\bar{x}^{B_w} \cap (\bar{H} \cup \bar{K}) = \bar{y}^{B_w} \cap (\bar{H} \cup \bar{K}) = \varnothing$. Hence, as before we can show that x, y are c.d.

This proves that if $\|x\| = \|y\| = 0$, then x, y are c.d.

Case 2. $\|x\| = \|y\| = \lambda \geqslant 1$. Recall that x, y are expressed as in 2.1. Suppose $x \not\sim_G y$. Clearly we can find w_0 such that $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$, $\bar{x}, \bar{y}$ being the images of x, y in G_{w_0}. By Collins' lemma we shall have $\bar{x} \sim_{G_{w_0}} \bar{y}$ if and only if $\bar{x} \sim_{\bar{H} \cup \bar{K}} \bar{y}^*$ where $\bar{y}^*$ is one of the cyclic permutations $\bar{t}^{f(i)}\bar{v}_i \cdots \bar{t}^{f(\lambda)}\bar{v}_\lambda \bar{t}^{f(1)}\bar{v}_1 \cdots \bar{t}^{f(i-1)}\bar{v}_{i-1}$ of $\bar{y}$. Our task is to find w_i so that, for each $\bar{z} \in \bar{H} \cup \bar{K}$, $\bar{x} \not\sim \bar{z}^{-1}(\bar{t}^{f(i)}\bar{v}_i \cdots \bar{t}^{f(i-1)}\bar{v}_{i-1})\bar{z}$ in G_{w_i}. For then, letting $w = w_1 \cdots w_\lambda$ we shall have $\bar{x} \not\sim_{G_w} \bar{y}$, whence x, y will be c.d. in G.

Without loss of generality, we show that there exists an integer w_1 such that $\bar{x} \not\sim_{G_{w_1}} \bar{z}^{-1}\bar{y}\bar{z}$ for each $\bar{z} \in \bar{H} \cup \bar{K}$. Now, except possibly in the case where $e(i) = f(i)$ $(i = 1, 2, \ldots, \lambda)$, we can choose $w_1 = w_0$. If now $e(i) = f(i)$ for each i then, by conditions (iii), (iv), (v) of Dyer [6, p. 39], $x \sim_G y$ if and only if there exists a complete set of solutions in B to Eqs. (4.3). Thus if $x \not\sim_G y$ we must have either (i) for some i, $u_i = W_i^{-1} v_i Z_i$ has no admissible solution in B or (ii) Eqs. (4.3) have admissible solutions ρ_i, σ_i for each i but do not have a complete set of solutions in B; that is, for some j $(j = 1, \ldots, \lambda)$ $\rho_j \neq t^{-e(j)}\sigma_{j-1}t^{e(j)}$ in B. (Recall σ_0 means σ_λ.) For case (i), by Lemmas 3.10

and 3.11, there exists an integer w_1^* such that $\bar{u}_i = \bar{W}_i^{-1}\bar{v}_i\bar{Z}_i$ has no admissible solution in $B_{w_1^*}$. Setting $w_1 = w_0 w_1^*$ we have $\bar{x} \not\sim \bar{y}$ in G_{w_1}, whence x, y are c.d. in G.

In case (ii) we assume Eqs. (4.3) have a complete set of solutions in some B_w. Consider the equation $u_i = W_i^{-1}v_iZ_i$ in B with solution ρ_i, σ_i. If ρ_i, σ_i are both from H then, by admissibility in B, we have $e(i) = -1$ and $e(i+1) = 1$. It follows, from the reducibility of x and y, that $u_i \notin H$ and $v_i \notin H$. Thus, by Lemma 4.3(ii), we see that ρ_i, σ_i is the unique admissible solution to the equation $u_i = W_i^{-1}v_iZ_i$. Then by Lemma 4.4(ii) there exists w, sufficiently large, such that, in B_w, the equation $\bar{u}_i = \bar{W}_i^{-1}\bar{v}_i\bar{Z}_i$ has the unique solution $\bar{\rho}_i, \bar{\sigma}_i$. Similar remarks apply if ρ_i, σ_i both lie in K.

Now suppose this unique system of admissible solutions to the equations $u_i = W_i^{-1}v_iZ_i$ in B is not complete. Then for some j ($j = 1,..., \lambda$) we have $\rho_j \neq t^{-e(j)}\sigma_{j-1}t^{e(j)}$ in B. This means, for example, that *if* $\rho_j = h^\alpha$ and $\sigma_{j-1} = k^\beta$ then $\alpha \neq \beta$. Of course, we may have, because of the lack of completeness, $\rho_j = h^\alpha$ and $\sigma_{j-1} = h^\beta$ with, possibly even $\alpha = \beta$. By the fact that the equations $u_i = W_i^{-1}v_iZ_i$ have unique admissible solutions in B and the equations $\bar{u}_i = \bar{W}_i^{-1}\bar{v}_i\bar{Z}_i$ have similar solutions in B_w it is clear that by re-choosing w sufficiently large one can find a B_w (or rather B_{ws}) in which the equations $\bar{u}_i = \bar{W}_i^{-1}\bar{v}_i\bar{Z}_i$ have admissible solutions but no complete set of solutions. Since by Lemma 3.13, B_{ws} is c.s., it follows that, by Theorem B, $\bar{x} \not\sim_{G_{ws}} \bar{y}$. Hence x, y are c.d. in G.

This completes the proof.

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